Introduction

This unit will help you build on what you have just learned about first moments to learn the very important skill of locating centroids.*

First it will deal with the centroids of simple geometric shapes. Then it will consider composite areas made up of such shapes.

As you progress in the study of mechanics you will find that you must locate many centroids quickly and accurately. In learning to do so you need little theory, but a great deal of practice is required.

Go to the next frame.

*If you have skipped Unit 11 do not be alarmed by the occasional calculus frame in this unit. Simply note the answer to such a frame, learn it as a given fact, and go on. It won't happen often.
Correct response to preceding frame

No Response

Frame 12-2

Definition

The centroid of an area is the point at which all the area could be concentrated without changing its first moment about any axis.

The "amoeba" shown at the left has an area of 3 cm².

If \( Q_x = 12 \text{ cm}^3 \) and \( Q_y = 9 \text{ cm}^3 \), we can use the definition above to locate the centroid. To find the vertical coordinate

\[
Q_x = A y \\
12 = 3 y \\
so \ y = 4
\]

Determine the horizontal coordinate of the centroid and supply the missing dimensions on the sketch.

\[
x = \underline{ } 
\]
Correct response to preceding frame

An area of $3 \text{ cm}^2$ "concentrated" at the dot would have a $Q_x = 3 \ (4)$ and a $Q_y = 3 \ (3)$.

Frame 12-3

**Definition**

The distance from an axis to the centroid is called "the centroidal distance."

In the figure point $G$ represents the centroid.

The centroidal distance from the $y$-axis is

________ in.

It represents the ____ coordinate of the centroid in the coordinate system shown.
Correct response to preceding frame

5 in. is the \( x \)-coordinate of the centroid.

Frame 12-4

**Nomenclature**

The centroidal distances found in this unit will be designated \( x_G \) and \( y_G \). The point representing the centroid will be labelled \( G \).

\( x_G \) is the ____ coordinate of the centroid and is the distance from the ____ axis.

Show \( G \) and \( x_G \) on the sketch.
$x_G$ is the x-coordinate and is the distance from the y-axis.

Frame 12-5

**Computing Centroidal Distances**

The distance from the centroid of a given area to a specified axis may be found by dividing the first moment of the area about the axis by the area.

For the area shown $A = 4 \text{ in}^2$. $Q_y = 16 \text{ in}^3$ and $Q_x = 18 \text{ in}^3$.

Find $x_G$ and $y_G$ and show $G$ by dimensioning the drawing.

$x_G = $ ____________________

$y_G = $ ____________________
Correct response to preceding frame

Frame 12-6

Computing Centroidal Distances

The first moments of the rectangle are $Q_x = 80 \text{ in}^3$ and $Q_y = 60 \text{ in}^3$.

$x_G = \text{______________}$

$y_G = \text{______________}$

Locate $G$ on the figure.
Correct response to preceding frame

\[ x_c = 3 \text{ in.}, \quad y_c = 4 \text{ in.} \]

Frame 12-7

**Transition**

By now you should have the general idea that the centroid is a point in the middle of the area and that you have to be able to find its co-ordinates.

In the next few frames we will briefly derive formulas for the centroidal coordinates for three simple shapes.

Rotate this paper rectangle about the axis along its left edge and read the next frame. (i.e. Turn the page!)
Frame 12-8

**Centroid of a rectangle**

The coordinate of the centroid of an area may be found by dividing the first moment of the area by the area thus

\[ y_G = \frac{Q_x}{A} \]

Set up an integral and find \( Q_x \).

Use it to find \( y_G \).

\[ y_G = \] _________________

What would you expect for \( x_G \)?

\[ x_G = \] _________________
Correct response to preceding frame

\[ Q_x = \int_{0}^{h} y \, dy = \frac{bh^2}{2} \]
\[ y_G = \frac{h}{2} \]
\[ x_G = \frac{b}{2} \]

**Solution:**

\[ Q_x = \int_{0}^{h} y \, dA = \int_{0}^{h} y \, b \, dy \]
\[ y_G = \frac{Q_x}{A} = \frac{\frac{bh^2}{2}}{bh} = \frac{h}{2} \]

---

Frame 12-9

**Centroid of a Rectangle**

Locate the centroid of the rectangle shown without integrating.
Correct response to preceding frame

\[ y_G = \frac{h}{2} = \frac{7}{2} \]

\[ x_G = \frac{b}{2} + 3'' = \frac{2}{2} + 3'' \]

Frame 12-10

**Centroid of a right triangle**

In the preceding unit you found the following first moments by integration. Use them to locate the centroid of the triangle.

\[ Q_x = \frac{bh^2}{6} \]

\[ Q_y = \frac{b^2h}{6} \]

\[ x_G = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \]  

\[ y_G = \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \]

Show your results on the figure.
Correct response to preceding frame

\[ x_G = \frac{b}{3} \]
\[ y_G = \frac{h}{3} \]

**Solution:**

\[ x_G = \frac{Q_y}{A} = \frac{b^2h}{6} \]
\[ y_G = \frac{Q_x}{A} = \frac{bh^2}{6} \]

---

**Frame 12-11**

**Centroid of a right triangle**

The centroid of a right triangle is shown as measured from the bases. What is its vertical distance from apex \( A \)?
Frame 12-12

**Centroid of a right triangle**

Locate the centroid of the right triangle shown.
Frame 12-13

Centroid of a Quarter Circle

You earlier found that for a quarter circle

\[ \rho_x = \frac{r^3}{3} \]

Locate the centroid on the drawing giving both coordinates.
Frame 12-14

Recapitulation

You now know the locations of the centroids of three important shapes. Summarize the preceding frames in your notebook on Page 12-1.
Frame 12-15

**Transition**

You have learned the vocabulary and notation for centroids and have learned to locate the centroids of three basic areas.

Using the areas you have learned plus a couple of simple rules it is often possible to locate the centroid of a figure simply by "eyeballing" its geometry.

The next several frames will be devoted to showing you these "tricks of the trade."

Go to the next frame.
Frame 12-16

**Centroids by Symmetry**

The centroid of any symmetrical area will fall on every axis of symmetry. Draw two axes of symmetry for each of the areas shown below to locate their centroids.
Correct response to preceding frame

Frame 12-17

**Centroids by Symmetry**

When an area has two or more axes of symmetry the centroid of the area will lie

________________________________________________________________________  .
Correct response to preceding frame

at the intersection of the axes of symmetry. (or equivalent response)

---

Frame 12-18

**Centroids by Symmetry**

Use symmetry to find the centroids of the areas shown below.

1)  
\[ \begin{align*}
  x_G &= \\
  y_G &= 
\end{align*} \]

2)  
\[ \begin{align*}
  x_G &= \\
  y_G &= 
\end{align*} \]
Correct response to preceding frame

(1.) $x_G = 6.00\text{ in.}$  \hspace{1cm} $y_G = 2.50\text{ in.}$
(2.) $x_G = 4.00\text{ cm}$  \hspace{1cm} $y_G = 0.00\text{ cm}$

Frame 12-19

Centroids by Symmetry

When an area has only one axis of symmetry only one coordinate of the centroid can be found by inspection.

Determine which of the centroidal coordinates can be found for the figures below and give their values.
Correct response to preceding frame

fig. (1) \( x_G = 2.50 \) cm  \hspace{1cm} fig. (2) \( x_G = 5.00 \) in.

Frame 12-20

**Centroids of Parts**

When the centroids of all parts of an area share a common centroidal coordinate, the centroid of the entire area will have that centroidal coordinate.

(1.) What is \( y_G \) for triangle A? (Refer to notebook if you need a hint.)

![Diagram of triangle A with labeled dimensions 4 in., 6 in., 6 in.]

(2.) What is \( y_G \) for triangle B?

![Diagram of triangle B with labeled dimensions 4 in., 6 in., 6 in.]

(3.) Locate the centroid of triangle C.

(4.) Give the coordinates of the centroid of triangle C.

\[
X_G = \underline{\underline{\phantom{9}}} \hspace{1cm} Y_G = \underline{\underline{\phantom{9}}}
\]
Correct response to preceding frame

(1.) \( y_G = 2 \text{ in.} \)
(2.) \( y_G = 2 \text{ in.} \)
(3.)

(4.) \( x_G = 0 \text{ in.} \) By symmetry \( y_G = 2 \text{ in.} \)

Frame 12-21

**Centroids from Parts**

Earlier you learned to locate the centroid of a quarter circle as shown.

Now locate the centroid of the semicircle shown below.
Correct response to preceding frame

\[ x_C = 8 + \frac{8}{\pi} = 10.6'' \]
\[ y_C = 10'' \]

Frame 12-22

Centroids from Parts

You know that for the right triangle shown

\[ x_C = \frac{b}{3} \quad \text{and} \quad y_C = \frac{h}{3} \]

Use this information to find one coordinate of the centroid of the scalene triangle below. (You cannot find the other as yet.) Name the dimensions as you need to.
Correct response to preceding frame

\[ y_G = \frac{h}{3} \]

**Solution:**
Divide the triangle into two right triangles. Locate their centroids, both at one-third the altitude and reason that the centroid of the entire triangle lies one-third the altitude above the base.

---

**Frame 12-23**

**Centroids from Parts**

Consider the scalene triangle below as being the difference of two right triangles.

Use what you know about right triangles to find one coordinate of the centroid of triangle A.

(Triangle A is formed by cutting triangle C away from triangle B.)
Correct response to preceding frame

For triangle A

\[ x_C = \frac{b}{3} \]

Frame 12-24

**Centroid of a triangle**

The centroid of any triangle is located ______________ of the ________ distance from any side to the opposite apex.
Correct response to preceding frame

**One-third** of the *perpendicular* distance

---

Frame 12-25

**Centroids**

Use what you have learned about triangles, symmetry, and plane trig to find the centroid of the equilateral triangle. A recommended first step is to locate the centroid on the drawing.
Correct response to preceding frame

\[ x_G = y_G = 5.6'' \]

**Solution:**

![Diagram showing a geometric figure with labeled dimensions and coordinates.]

\[ x_G = a + b \cos 45^\circ \]

\[ a = 5 \cos 45^\circ = 5 (.707) \]

\[ b = \frac{1}{3} \sin 60^\circ \]

\[ x_G = 5(.707) + \frac{8.66}{3} (.707) \]

---

Frame 12-26

**Transition**

You should now be pretty proficient at locating centroids by inspection. Don’t neglect the skill. Properly encouraged, it can save you much time and computation.

Even skillful inspection, however, can take you only so far. The remainder of this unit might be called "What to do after the inspector gives up." and will teach you to solve problems the long, hard, foolproof way—and really it’s not so hard.

Go to the next frame.
The basic definition for first moments that we have been using is that the first moment of an area about a given axis is equal to the sum of the first moments of all parts of the area about the axis.

Thus, for the area shown the total first moment is:

\[ Q_{x\text{ total}} = Q_{x1} + Q_{x2} \]

and the total area is

\[ A_{\text{total}} = A_1 + A_2 \]

The y-coordinate of the centroid is thus:

\[ y_G = \frac{Q_{x\text{ total}}}{A_{\text{total}}} = \]
The figure shows a trapezoid broken into a triangle and a rectangle. The table below shows a systematic way of finding the centroid of such a composite area. It is completed for $x_G$ only.

Finish it to find $y_G$.

($A_p$ means "Area of the part" etc.)
<table>
<thead>
<tr>
<th>Shape</th>
<th>$A_p$</th>
<th>$x_{Gp}$</th>
<th>$A_p x_{Gp}$</th>
<th>$y_{Gp}$</th>
<th>$A_p y_{Gp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>9</td>
<td>4</td>
<td>36</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>48</td>
<td>3</td>
<td>144</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Totals:</td>
<td>57</td>
<td></td>
<td>180</td>
<td>$Q_y$</td>
<td>$Q_x$</td>
</tr>
</tbody>
</table>

$$x_G = \frac{Q_y}{A} = \frac{180}{57} = 3.16 \text{ in.}$$

$$y_G = \_\_\_\_\_\_$$
Correct response to preceding frame

\[ y_G = 4.8" \]

<table>
<thead>
<tr>
<th>Shape</th>
<th>( A_p )</th>
<th>( y_{Gp} )</th>
<th>( A_p y_{Gp} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \triangle )</td>
<td>9</td>
<td>9</td>
<td>81</td>
</tr>
<tr>
<td>( \square )</td>
<td>48</td>
<td>4</td>
<td>192</td>
</tr>
<tr>
<td>Total</td>
<td>57</td>
<td>( \frac{273}{57} )</td>
<td></td>
</tr>
</tbody>
</table>

\[ y_G = \frac{\sum x}{A} = \frac{273}{57} \]

Frame 12-29

Composite Areas

Use the table below to find \( y_G \) for the figure shown.

<table>
<thead>
<tr>
<th>Shape</th>
<th>( A_p )</th>
<th>( y_{Gp} )</th>
<th>( A_p y_{Gp} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \triangle )</td>
<td>( \square )</td>
<td>( \text{Totals:} )</td>
<td></td>
</tr>
</tbody>
</table>

\[ y_G = \underline{\text{(value)}} \]

Now find \( x_G \)

\[ x_G = \underline{\text{(value)}} \]
Correct response to preceding frame

\[ y_G = 6.67 \text{ in.} \]
\[ x_G = 6.00 \text{ in.} \]

<table>
<thead>
<tr>
<th>Shape</th>
<th>( A_p )</th>
<th>( y_{Gp} )</th>
<th>( A_p y_{Gp} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>△</td>
<td>18</td>
<td>10</td>
<td>180</td>
</tr>
<tr>
<td>□</td>
<td>36</td>
<td>5</td>
<td>180</td>
</tr>
<tr>
<td>Totals</td>
<td>54</td>
<td></td>
<td>360</td>
</tr>
</tbody>
</table>

\[ A = 54 \quad Q_x = 360 \]
\[ y_G = \frac{Q_x}{A} = \frac{360}{54} = 6.67 \text{ in.} \]
\[ x_G = 6 \text{ in. by inspection} \]

Frame 12-30

**Signs**

Sometimes the centroid of part of an area is so located as to have a negative coordinate. In such a case the first moment will be negative and the centroid of the entire area may have a negative coordinate. Complete the following computation and locate the centroid on the sketch.
<table>
<thead>
<tr>
<th>Shape</th>
<th>$A_p$</th>
<th>$x_{Gp}$</th>
<th>$A_p x_{Gp}$</th>
<th>$y_{Gp}$</th>
<th>$A_p y_{Gp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>□</td>
<td>24</td>
<td>3</td>
<td>72</td>
<td>-2</td>
<td>-48</td>
</tr>
</tbody>
</table>
Correct response to preceding frame

\[ x_G = 4.08 \text{ cm} \]
\[ y_G = -2.16 \text{ cm} \]

<table>
<thead>
<tr>
<th>Shape</th>
<th>( A_p )</th>
<th>( x_p )</th>
<th>( A \times x_p )</th>
<th>( y_p )</th>
<th>( A_p \times y_p )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>24</td>
<td>3</td>
<td>72</td>
<td>-2</td>
<td>-48</td>
</tr>
<tr>
<td></td>
<td>8</td>
<td>7.33</td>
<td>58.67</td>
<td>-( \frac{8}{3} )</td>
<td>-21.3</td>
</tr>
<tr>
<td>Totals</td>
<td>32</td>
<td></td>
<td>130.67</td>
<td>-69.3</td>
<td></td>
</tr>
</tbody>
</table>

Frame 12-31

**Composite Areas**

It is sometimes convenient to work with reference axes which intersect the area, even though this results in some negative coordinates. Compare the two partial solutions below and complete the one that seems easier.

\[ x_G = 2 \text{ by inspection} \]

<table>
<thead>
<tr>
<th>Shape</th>
<th>( A_p )</th>
<th>( y_{Gp} )</th>
<th>( A_p \times y_{Gp} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bigodot )</td>
<td>( 2\pi )</td>
<td>( 6 + \frac{8}{3\pi} )</td>
<td></td>
</tr>
<tr>
<td>( \bigtriangleup )</td>
<td>12</td>
<td>4</td>
<td></td>
</tr>
</tbody>
</table>

\[ x_G = 0 \text{ by inspection} \]

<table>
<thead>
<tr>
<th>Shape</th>
<th>( A_p )</th>
<th>( y_{Gp} )</th>
<th>( A_p \times y_{Gp} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bigodot )</td>
<td>( 2\pi )</td>
<td>( \frac{8}{3\pi} )</td>
<td></td>
</tr>
<tr>
<td>( \bigtriangleup )</td>
<td>12</td>
<td>-2</td>
<td></td>
</tr>
</tbody>
</table>
Correct response to preceding frame

Frame 12-32

Composite Areas

Using the indicated axes as references, find the coordinates of the centroid of the area shown.
<table>
<thead>
<tr>
<th>Part</th>
<th>$A_p$</th>
<th>$x_{Gp}$</th>
<th>$A_p x_{Gp}$</th>
<th>$y_{Gp}$</th>
<th>$A_p y_{Gp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>△</td>
<td>6</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>▽</td>
<td>4.5</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>■</td>
<td>12</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$A_{total} =$

$Q_x =$  

$y_G =$

$Q_y =$  

$x_G =$
Correct response to preceding frame

\[ x_G = 1.22 \text{ m} \]
\[ y_G = -0.725 \text{ m} \]

<table>
<thead>
<tr>
<th>Part</th>
<th>( A_p )</th>
<th>( x_{Gp} )</th>
<th>( A_p x_{Gp} )</th>
<th>( y_{Gp} )</th>
<th>( A_p y_{Gp} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \triangle )</td>
<td>6</td>
<td>( \frac{4}{3} )</td>
<td>8</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>( \square )</td>
<td>4.5</td>
<td>-1</td>
<td>-4.5</td>
<td>-1</td>
<td>-4.5</td>
</tr>
<tr>
<td>( \square )</td>
<td>12</td>
<td>2</td>
<td>24</td>
<td>-1.5</td>
<td>-18</td>
</tr>
</tbody>
</table>

\[ A_{total} = 22.5 \]
\[ Q_y = 27.5 \]
\[ Q_x = -16.5 \]

Frame 12-33

Negative Areas

Some areas have holes in them. Handle a hole as a negative area. Complete the problem below. Find the centroid of the cross-hatched area.

\[ \begin{align*}
A_{total} &= \_\_\_\_\_\_ \\
Q_y &= 20.86 \\
x_G &= 2 \text{ m} \\
y_G &= \_\_\_\_\_\_ \\
\end{align*} \]
Correct response to preceding frame

$Q_x = 17.33$

$y_G = 1.66 \text{ m}$

<table>
<thead>
<tr>
<th>$A_p$</th>
<th>$y_{Gp}$</th>
<th>$A_p y_{Gp}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>1.5</td>
<td>18</td>
</tr>
<tr>
<td>$-\frac{\pi}{2}$</td>
<td>$\frac{4}{3\pi}$</td>
<td>$-0.67$</td>
</tr>
</tbody>
</table>

Frame 12-34

**Negative Areas**

Find the centroid of the area shown.

(Practice drawing your own table from memory.)
Correct response to preceding frame

\[ x_G = 2.39 \text{ in.} \]
\[ y_G = 2.00 \text{ in. by symmetry} \]

<table>
<thead>
<tr>
<th>Part</th>
<th>( A_p )</th>
<th>( x_{Gp} )</th>
<th>( A_p x_{Gp} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \square )</td>
<td>12</td>
<td>( \frac{3}{2} )</td>
<td>18</td>
</tr>
<tr>
<td>( \bigcirc )</td>
<td>6.28</td>
<td>3.85</td>
<td>24.2</td>
</tr>
<tr>
<td>( \square )</td>
<td>-4</td>
<td>2</td>
<td>-8</td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td>14.28</td>
<td></td>
<td>34.2</td>
</tr>
</tbody>
</table>

Frame 12-35

**Negative Areas**

Same area as last frame. Different reference axes. Find \( x_G \) using the axes shown. Make sure you get the same location (not coordinate) you got before.
Correct response to preceding frame

\[ x_G = -0.61 \text{ in.} \]

<table>
<thead>
<tr>
<th>Part</th>
<th>( A_p )</th>
<th>( x_{Gp} )</th>
<th>( A_p x_{Gp} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>12</td>
<td>(-\frac{3}{2})</td>
<td>-18</td>
</tr>
<tr>
<td>2</td>
<td>6.28</td>
<td>(\frac{8}{3\pi})</td>
<td>5.33</td>
</tr>
<tr>
<td>3</td>
<td>-4</td>
<td>-1</td>
<td>+4</td>
</tr>
<tr>
<td>Totals</td>
<td>14.28</td>
<td></td>
<td>-8.67</td>
</tr>
</tbody>
</table>

Frame 12-36

**Selection of Reference Axes**

There is no one right way to select reference axes. One selection may give you all positive numbers, but bad arithmetic. Another may clean up the arithmetic at the cost of negative signs. Each time you work a problem you must make a choice. The thing to remember is: **choose** and get to work. Think about the advantages and disadvantages of each axis location before you begin work.

The centroid isn’t “just math,” it’s a physical location. I usually provide the answer on a sketch so you can quickly check your work. I would suggest that you create a sketch with your work so that you can check whether your looks right, and to make it easier for a grader to do so.

This last set of problems will require you to select your own reference axes, but whatever your choice, the actual location of the centroid should agree with the answer given.

Go to the next frame.
Correct response to preceding frame

No response

Frame 12-37

**Centroid of a Composite Area**

Find the centroid of the area shown and locate it on a sketch.
Correct response to preceding frame

Frame 12-38

Centroid of a Composite Area

Locate on a sketch the centroid of the area shown.
Correct response to preceding frame

![Diagram of a composite area with labeled centroid]

<table>
<thead>
<tr>
<th>Part</th>
<th>$A_p$</th>
<th>$y_{Gp}$</th>
<th>$A_p \times G_p$</th>
</tr>
</thead>
<tbody>
<tr>
<td>⊙</td>
<td>-6.28</td>
<td>$\frac{8}{3\pi}$</td>
<td>-5.33</td>
</tr>
<tr>
<td>□</td>
<td>15</td>
<td>$-\frac{3}{2}$</td>
<td>-22.5</td>
</tr>
<tr>
<td>△</td>
<td>12.5</td>
<td>$\frac{5}{3}$</td>
<td>20.8</td>
</tr>
<tr>
<td>Totals</td>
<td>21.22</td>
<td>–</td>
<td>-7.03</td>
</tr>
</tbody>
</table>

$y_G = -0.331$

Frame 12-39

**Centroid of a Composite Area**

Find the centroid of the area shown and locate it on a sketch.
Frame 12-40

Summary

You will be happy to know that the end is near. Fill in Page 12-2 in the notebook and read the closure, and you’re done!
Frame 12-41

Closure

If you have been moderately successful in calculating the centroids in this unit you should be able to locate the centroid of any area. If you can break it down into simple geometric shapes you can use composite methods directly. If some parts require integration, you must do that first then use composite methods. In mechanics of materials you will learn to combine areas of cross-sections for composite areas made up of "steel beams" and other shapes you will look up in tables.

Given sufficient time and patience, you will always win through to a right answer eventually Well, almost always.