Frame 30-1

**Introduction**

This unit will deal with the moment of inertia of mass, which should hardly be a new concept to you, since you have encountered it previously in math, and has many similarities to moment of inertia of area. We will therefore pass rather lightly over its more mathematical aspects and concentrate instead on the actual arithmetical juggling involved in finding the moments of inertia of both simple geometric shapes and composite bodies.

Like moment of inertia of area, we might argue that "this isn’t statics," but it is usually covered in statics courses as preparation for dynamics. The basis for this is that, to work problems dealing with rotation and plane motion of bodies, it is mandatory that we either have, or be able to determine, correct values for the moments of inertia of masses.

The first section of this unit will summarize the mathematical concept of moment of inertia of mass.

Go to the next frame.
By definition, the moment of inertia of an element of mass $dM$ with respect to any plane (or axis) is the product of the mass of the element and the "square-of-the-distance" from the element to the plane (or axis).*

So we see that:

![Diagram of a three-dimensional object with a mass element $dM$ highlighted.

- About the $xz$-plane, $dI_{xz} = y^2 \, dM$
- About the $x$-axis, $dI_x = (y^2 + z^2) \, dM$

(a.) About the $xy$-plane, $dI_{xy} = \quad$ ____________

(b.) About the $yz$-plane, $dI_{yz} = \quad$ ____________

(c.) About the $y$-axis, $dI_y = \quad$ ____________

(d.) About the $z$-axis, $dI_z = \quad$ ____________

*Comment: Don’t worry about the physical significance of a moment of inertia about a plane -- it hasn’t any; it is merely a mathematical artifice!
Correct response to preceding frame

(a.) About the xy-plane, \( dI_{xy} = z^2 \, dM \)
(b.) About the yz-plane, \( dI_{xz} = x^2 \, dM \)
(c.) About the y-axis, \( dI_y = (x^2 + z^2) \, dM \)
(d.) About the z-axis, \( dI_z = (x^2 + y^2) \, dM \)

Frame 30-3

Review

You have previously learned to define the moment of inertia of an area with respect to a given axis (for example, the x-axis) as

\[
I_x = \int y^2 \, dA
\]

The moment of inertia of a mass with respect to the same axis is

\[
I_x = \int \rho^2 \, dM
\]

In both cases, however, the element must be chosen so that the entire element is the same ______________ from the x-axis.
Frame 30-4

Choice of Element

Check those cases in which the element is properly chosen for the corresponding integral.
(a) \[ I_x = \int y^2 \, dA \]

(b) \[ I_x = \int y^2 \, dA \]

(c) \[ I_x = \int y^2 \, dA \]

(d) \[ I_x = \int \rho^2 \, dM \]

(e) \[ I_x = \int \rho^2 \, dM \]

(f) \[ I_x = \int \rho^2 \, dM \]
Correct response to preceding frame

(a), (b), (f) In the other cases all parts of the element are not equidistant from the x-axis.

Frame 30-5

**Choice of Element**

Sketch two elements of mass which might be used to find the moment of inertia of the cone about the y-axis by using

\[ I_y = \int r^2 \, dM \]

Sketch two elements which could be used to find the moment of inertia of the paraboloid about the x-z plane by using

\[ I_{xz} = \int y^2 \, dM \]
Limits

After a proper element of mass has been selected, the next step is to establish the proper limits for the integral \( I = \int \rho^2 \, dM \).

The paraboloid shown is composed of a homogeneous material of density \( \delta \).

The element mass is then given by

\[
dM = \delta \, dv = \delta \, dx \, dy \, dz
\]

and the square of its distance from the \( z \)-axis will be

\[
\rho^2 = x^2 + y^2
\]

The moment of inertia about the \( z \)-axis will thus be

\[
I_z = \int \int \int \delta \left[ x^2 + y^2 \right] \, dx \, dy \, dz
\]

Supply the correct limits for the integral exactly as given.
Correct response to preceding frame

\[
I_z = \left\{ \begin{array}{c}
\int_{-a\sqrt{z/h}}^{a\sqrt{z/h}} \int_0^{+\sqrt{\frac{a^2}{h} - y^2}} \int_{\sqrt{\frac{a^2}{h} - y^2}}^{+\sqrt{\frac{a^2}{h} - y^2}} \delta \left[ x^2 + y^2 \right] dx \, dy \, dz \\
0 & & \end{array} \right. 
\]

If your limits are correct, skip to Frame 30-12; if not, go to the next frame.

Frame 30-7

**Limits**

In integrating \( \int \int \int dx \, dy \, dz \) with respect first to \( x \), we are simply expanding the element in the "\( x \)-direction." The limits are the locations where the expanding element pierces the surface of the volume.

1. Show the element expanded in the \( x \)-direction.

2. Which of the following are the proper limits?

   (a) \( \int_{-a}^{a} \)
   (b) \( \int_{-f(y,z)}^{f(y,z)} \)
The variable radius of the small circle can be expressed in terms of $z$ as

$$r = \sqrt{z/k}$$

By means of the Pythagorean theorem, find $f(y, z)$ (defined on the lower drawing).

$$f(y, z) = \text{______________}$$
Correct response to preceding frame

\[ f(y, z) = \sqrt{\frac{z^2}{k} - y^2} = \sqrt{\frac{z}{k} - y^2} \]

Frame 30-9

**Limits**

The result of the integration of \( dx \) in

\[
\int \int \int_{-\sqrt{\frac{z}{k} - y^2}}^{\sqrt{\frac{z}{k} - y^2}} dx \ dy \ dz
\]

was to let the differential cube expand in the \( x \)-direction into a rod.

The next integration will let the rod expand in the \( y \)-direction until it is again limited by the "surface."

1. Draw the resulting element.

2. The correct limits on \( y \) are

   (a) \[ \int_{-a}^{a} \]  
   (b) \[ \int_{-g(z)}^{g(z)} \]
Correct response to preceding frame

1. 

2. 

The correct limits are b)\[ \int_{-g(z)}^{g(z)} \]

Since the element is not at the top, its radius is governed by its position along the z-axis.

Frame 30-10

Limits

Supply the limits for the integration of dy in

\[ \int_{-\sqrt{z/k}-y^2}^{\sqrt{z/k}-y^2} \int_{-\sqrt{z/k}-y^2}^{\sqrt{z/k}-y^2} dx \, dy \, dz \]

for the figure shown.
The final integration consists of stacking up the disks obtained by the two previous integrations until the paraboloid is filled in.

Complete the limits.

\[
\int_{-\sqrt{z/k}}^{\sqrt{z/k}} \int_{-\sqrt{(z/k) - y^2}}^{\sqrt{(z/k) - y^2}} dx \, dy \, dz
\]
The height of the paraboloid, $h$, is equal to $ka^2$.

Frame 30-12

Limits

The method you have just completed -- that of visualizing the expansion of the element due to each integration in turn -- may be used on any multiple integration problem and will reduce the problem of finding limits to a problem in geometry. A good choice for the order of integration results in less work than a poor choice. Consequently, you may actually save time by sketching elements for more than one order and choosing the order that looks easiest.

Go to the next frame.
The methods of multiple integration may be well known to you, and you may have a few tricks of your own for shortening the work. If you want a more thorough treatment, you will find it in the nearest text on integral calculus; complete with a nice (?) selection of problems. You will not need it to complete this unit.

Anytime you need to find the moment of inertia of a mass bounded by complicated mathematical surfaces, multiple integration is the tool you must use. Most engineering problems, however, deal with bodies which can be built up out of simple geometric shapes. The next section of this unit will deal exclusively with these shapes. It will take you roughly 60 minutes to reach the next transition, so take a short break before going to the next frame.

When you return, GO-GO--GO!
The simplest geometric form is a rectangular parallelepiped.

Assume the parallelepiped shown to be of constant density $\delta$. Now let’s test your mathematical virtuosity by finding its moment of inertia about the $x$-axis, which just happens to pass through the mass-center of the body. (As also do the $y$ and $z$ axes.)

(a.) Make a sketch of an appropriate element.

(b.) Set up your integral, but do not integrate yet.
Frame 30-15

Geometric Forms

Now that you have the appropriate integral expression for $I_x$, namely

$$I_x = \int_{c/2}^{b/2} \int_{a/2}^{c/2} \int_{-b/2}^{-a/2} \delta \left( y^2 + z^2 \right) dx \, dy \, dz$$

Complete the integration.

$I_x =$ ____________________________
Correct response to preceding frame

\[ I_x = \frac{\delta ab^3 c + \delta abc^3}{12} \]

**Solution:**

\[
I_x = \delta \left[ \frac{b}{2} \int_{-b/2}^{b/2} \left( y^2 + z^2 \right) \, dy \right]_{-a/2}^{a/2}
\]

\[
= \delta \left[ \frac{c}{2} \int_{-c/2}^{c/2} \left( ay^2 + az^2 \right) \, dy \right]_{-b/2}^{b/2}
\]

\[
= \delta \left[ \frac{c}{2} \left( \frac{ay^3}{3} + az^2 y \right) \bigg|_{-b/2}^{b/2} \right]_{-c/2}^{c/2}
\]

\[
= \delta \left[ \frac{c}{2} \left( \frac{ab^3}{12} + abz^2 \right) \right]_{-c/2}^{+b/2}
\]

\[
= \delta \left[ \frac{ab^3 c + \delta abc^3}{12} \right]_{-c/2}^{c/2}
\]

Frame 30-16

**Geometric Forms**

For a homogeneous parallelepiped

\[ I_x = \frac{\delta ab^3 c + abc^3}{12} \]

\[ = \frac{\delta abc}{12} \left( b^2 + c^2 \right) \]

What is the mass of the body?

\[ M = \text{______________} \]

Write \( I_x \) in terms of the mass.

\[ I_x = \text{______________} \]
Correct response to preceding frame

\[ M = \delta abc \]

\[ I_x = \frac{M}{12} \left[ b^2 + c^2 \right] \]

---

Frame 30-17

**Geometric Forms**

For the homogeneous mass shown

\[ I_x = \frac{M}{12} \left[ b^2 + c^2 \right] \]

---

Does the dimension parallel to the x-axis appear in the expression for \( I_x \)?

[ ] Yes  [ ] No

Without integrating, write analogous expressions for \( I_y \) and \( I_z \); assuming the z and y-axes pass through the mass center.

\[ I_y = \__________________________ \]

\[ I_z = \__________________________ \]
Another easy and important geometric form is the homogeneous cylinder. For this body though, it’s easier to use cylindrical coordinates. To find $I_z$ (the moment about the geometrical axis) consider an element of mass at a distance $\rho$ from the $z$-axis.

The mass element will be given by

$$dM = (\rho d\theta)(d\rho)(dz).$$

Plugging this into

$$I_z = \int \int \int \rho^2 dM$$

we have

$$I_z = \int \int \int \delta \rho^3 d\theta d\rho dz$$

Supply the limits for $z$, $\theta$, and $\rho$, and complete the integration.
Geometric Forms

For the homogeneous cylinder we have

\[ I_z = \frac{1}{2} \pi r^4 h \]

Determine the mass of the cylinder and write its moment of inertia about its geometric axis in terms of its mass.

\[ M = \text{________________________} \]

\[ I_z = \text{________________________} \]
Correct response to preceding frame

\[ M = \delta \pi r^2 h \]

\[ I_z = \frac{Mr^2}{2} \]

---

Frame 30-20

**Geometric Forms**

Many engineering parts are basically cylindrical because cylinders are easy to turn on a lathe, or to roll from sheets. On the previous two frames we found the moment about the geometric-axis. We will now turn our attention to finding the moment of inertia of the cylindrical mass about a diametric-axis.

Assuming the origin is taken at the mass center, set up the integral necessary to find \( I_x \). (Cylindrical coordinates again are easiest.)
\[ I_x = \delta \left\{ \quad \right\} \]
Correct response to preceding frame

\[ I_x = \delta \int_{-h/2}^{h/2} \int_0^{2\pi} \int_0^r \left( \rho^2 \sin^2 \theta + z^2 \right) \rho \, d\rho \, d\theta \, dz \]

Using cylindrical coordinates \( dM = \delta \rho \, d\rho \, d\theta \, dz \) and is located \( \sqrt{\rho^2 \sin^2 \theta + z^2} \) from the x-axis.

---

**Geometric Forms**

Evaluate the integral you obtained in the preceding frame, and express your answer in terms of mass.
For the homogeneous right circular cylinder shown, the moment of inertia about a diametric axis through the mass center is:

\[ I_x = \frac{Mr^2}{4} + \frac{Mh^2}{12} \]

Suppose you have a cylinder with a radius that is very large compared to its height. Write an approximate value for \( I_x \) for such a cylinder.

\[ I_x = \text{_______________} \]

Compare the result with the expression found on Frame 30-19.

Why is there a difference? _____________________________________________________
Correct response to preceding frame

\[ I_x = \frac{Mr^2}{4} \]

This is the moment of inertia of a thin disk about a diametric axis such as D-D, where as

\[ I_z = \frac{Mr^2}{2} \]

(obtained from Frame 30-19) is the moment of inertia about its geometric axis z.

Frame 30-23

Geometric Shapes

For the right circular cylinder shown, the moment of inertia about a diametric axis through the mass center is

\[ I_x = \frac{Mr^2}{4} + \frac{Mh^2}{12} \]

Write an approximation for the diametric moment of inertia of a cylinder which has a radius much smaller than its height.

\[ I_x = \_______________ \]

For the parallelepiped shown, the moment of inertia about an x-axis through the mass center and parallel to one of the longitudinal surfaces is
Write an approximation for the perpendicular moment of inertia if "b" is very small compared to "h".

\[ I_x = \frac{Mb^2}{12} + \frac{Mh^2}{12} \]

1. What shape results if "b" is small compared to "h" but "a" is not? ______________

2. Would the same approximation for Ix be appropriate in such a case?

☐ Yes  ☐ No
Correct response to preceding frame

\[ I_x = \frac{Mh^2}{12} \] which is correct for a slender cylindrical rod with \( M = \delta \pi r^2 h \)

\[ I_x = \frac{Mh^2}{12} \] which is correct for a slender rod with a rectangular cross-section and \( M = \delta abh \)

1. A plane, thin plate, or sheet
2. Yes

Frame 30-24

**Geometric Shapes**

The shape of the cross-section of a rod has a negligible effect on its moment of inertia

if \( \text{______________________________} \).
Correct response to preceding frame

the rod is slender enough

Frame 30-25

**Notebook**

Complete Page 30-1 and Page 30-2 in your notebook.
Correct response to preceding frame

\[
\frac{Mr^2}{2} \\
\frac{Mr^2}{4} \\
\frac{Mc^2}{4} + \frac{Mb^2}{12} \\
\frac{Mb^2}{12} + \frac{Mc^2}{12} \\
\frac{Ml^2}{12}
\]

(cylinder about its geometric axis)

(thin disk about a diameter)

(cylinder about a diameter through the mass center)

(rectangular parallelepiped about axis through the mass center and parallel to an edge)

(slender rod about an axis through the mass center and parallel to longitudinal axis)

Frame 30-26

**Transition**

The moments of inertia of the forms you have just studied -- and many more -- will be found in many reference books of mathematical tables, engineering handbooks, and in many texts. Although you can generally look up the moment of inertia of simple geometric shapes, you may find it convenient to memorize a few of the most common ones (at least for exams!)

The simple shapes can be combined to form many composite bodies. This, however, requires that you be familiar with the parallel axis theorem for masses. The next section of this unit will be devoted to that theorem and its application.

Go to the next frame.
Parallel Axis Theorem

The figure shows the cross-section of a body containing the mass center, G. The element \( dM \) is at a distance \( \sqrt{(d + x)^2 + y^2} \) from the axis \( z' \). The moment of inertia about the \( z' \)-axis is therefore

\[
I_{z'} = \int [(x + d)^2 + y^2] \, dM
\]

which can be expanded and rewritten as

\[
I_{z'} = \int (x^2 + y^2) \, dM + 2d \int x \, dM
\]

From this it is observed that

\[
\int (x^2 + y^2) \, dM = \quad \text{__________}
\]

and upon integrating, that

\[
d^2 \int dM = \quad \text{______}
\]

"\( x \)" is measured from an axis through the mass center and the first moment of any mass about its own center is \( \text{________________________} \) / 

Therefore,

\[
2d \int x \, dM = \quad \text{______}
\]
Correct response to preceding frame

\[ \int \left[ x^2 + y^2 \right] dM = I_z \]
\[ d^2 \int dM = Md^2 \]
zero
\[ 2d \int xdM = 0 \]

Frame 30-28

**Parallel-Axis Theorem**

\[ I_{z'} = \int \left[ x^2 + y^2 \right] dM + d^2 \int dM + 2d \int x dM \]

Use the equalities obtained in the preceding frame to rewrite \( I_z \), in terms of \( I' \)
\[ I_{z'} = \]
I_{z'} = I_z + Md^2 or I_{z'} = I_{zG} + Md^2 (That looks a lot like the parallel axis theorem for moment of inertia of area, I_{z'} = I_{zG} + Ad^2, doesn’t it?)

Parallel Axis Theorem

The parallel axis theorem states that the moment of inertia of a body about any axis is equal to the moment of inertia of the body with respect to a parallel axis through the mass center of the body plus the product of the mass of the body and the square of the distance between axes.

In equation form this statement becomes \( l_{xa} = l_{xG} + Md^2 \) where:

1. \( l_{xa} \) is ___________________________________________________
2. \( l_{xG} \) is ___________________________________________________
3. \( d \) is _____________________________________________________
In \( l_{x0} = l_{xG} + Md^2 \)
1. \( l_{x0} \) is the moment of inertia about any axis.
2. \( l_{xG} \) is the moment of inertia about a parallel axis through the mass center.
3. \( d \) is the distance between the axes.

Frame 30-30

**Parallel Axis Theorem**

In each case the moment of inertia about axis A is known. Is it possible to find the moment of inertia about B by means of the parallel axis theorem? If your answer is "no", explain why.

1. (1.)
   - Yes
   - No
   Why not? ________________________________
   ________________________________

2. (2.)
   - Yes
   - No
   Why not? ________________________________
   ________________________________

3. (3.)
   - Yes
   - No
   Why not? ________________________________
   ________________________________

4. (4.)
   - Yes
   - No
   Why not? ________________________________
   ________________________________
Correct response to preceding frame

1. Yes
2. No. A is not parallel to B.
3. Yes
4. No. Neither A nor B is through the mass center.

Frame 30-31

Units

The units of mass in the American Customary Units system are slugs.

A slug is a mass which weighs approximately 32 pounds on the surface of the earth. You can worry about why this is true when you study dynamics, but it's even more natural than having twelve inches to a foot.

1. Convert the following:

   32 lb = ________________ slugs

   160 lb = ________________

   48 lb = ________________

   9.6 lb = ________________

2. Moment of inertia of mass is the product of mass and the square of a distance.

   (a.) In ACU, moment of inertia of mass is expressed in ________________ .

   (b.) In SI, mass is expressed in ________________ , and moment of inertia of mass is expressed in ________________ .
Correct response to preceding frame

32 lb = 1 slug
160 lb = 5 slugs
48 lb = 1.5 slugs
9.6 lb = 0.3 slug

2. (a.) slug-ft^2 (slug-in^2 might also be used, but it is not common.)
(b.) kilograms kg-m^2

Frame 30-32

**Parallel Axis Theorem**

For the homogeneous disk shown

- M = 3 kilograms
- r = 2 meters
- d = 5 meters

Find \( I_B \).

\[ I_B = \underline{\text{_______________________}} \]
Correct response to preceding frame

\[ I_B = \frac{1}{2} M r^2 + M d^2 \]

\[ = \frac{1}{2} \times 3 \times (4) + 3 \times (25) \]

\[ = 81 \text{ kg-m}^2 \]

Frame 30-33

**Parallel Axis Theorem**

The mass of the body shown is 2 slugs.

Find \( I_B \).

\[ I_B = \text{______________} \]
The preceding sections of this unit have dealt with moments of inertia of masses by integration, the moments of inertia of simple geometric forms, and the parallel axis theorem for masses.

The next section will show you how to find the moments of inertia of various complex mass forms which are built up by combining simpler shapes. You will, however, need either to be familiar with the expressions for the moments of inertia of various simple bodies or be able to look them up in a set of tables.

Although the upcoming section is relatively short, you'd probably better take a short breather here.
Each rod weighs 20 kilograms and is 30 centimeters ft long.

1. What is the moment of inertia of the vertical rod about an axis through its mass center parallel to the A-A axis?
   \[ I_{GA} = \] _________________

2. What is the moment of inertia of the vertical rod about the A-A axis?
   \[ I_{A1} = \] _________________

3. What is the moment of inertia of the horizontal rod about an axis through its mass center parallel to A-A?
   \[ I_{GA} = \] _________________

4. What is the moment of inertia of the horizontal rod about A-A?
   \[ I_{A2} = \] _________________

5. What is the moment of inertia of the composite body about A-A?
   \[ I_{A} = \] _________________
Composite Bodies

For complex bodies a tabular computation can be very helpful.

Complete the following table.

All bodies are solid and homogeneous. A and C are spherical.

- A weighs 16 pounds
- B weighs 4 pounds
- C weighs 8 pounds

The term $y_{PG}$ is the distance from the mass center of the part to the A-axis. Remember to get your mass by dividing weight by the constant 32 fps$^2$. 

For vertical rod

1. $0.150 \text{ kg-m}^2$

\[
I_G = \frac{1}{12} Ml^2 = \frac{1}{12} (20)(0.30)^2
\]

2. 1.95

\[
I_{A1} = I_G + Md^2
= 0.150 + (20)(0.30)^2
\]

For horizontal rod

3. $0.150 \text{ kg-m}^2$

\[
I_G = \frac{1}{12} Ml^2 = \frac{1}{12} (20)(0.30)^2
\]

4. $0.600 \text{ kg-m}^2$

\[
I_{A2} = I_G + Md^2
= 0.150 + (20)\left[\frac{0.30}{2}\right]^2
\]

For both rods

5. $2.55 \text{ kg-m}^2 = I_{A1} + I_{A2}$
<table>
<thead>
<tr>
<th>Body</th>
<th>Mass</th>
<th>$y_{GP}$</th>
<th>$M_{y_{GP}}$</th>
<th>$M_{y_{GP}}^2$</th>
<th>$I_{AGP}$</th>
<th>$I_{AP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\frac{1}{2}$</td>
<td>8</td>
<td>4</td>
<td>32</td>
<td>$\frac{2}{5} M r^2 = \frac{2}{5} \frac{1}{2} (4) = \frac{4}{5}$</td>
<td>32.8</td>
</tr>
<tr>
<td>B</td>
<td>$\frac{1}{8}$</td>
<td>3</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{9}{8}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$I_A =$ ________________
Correct response to preceding frame

$$I_A = 34.65 \text{ slug-ft}^2$$

Frame 30-37

**Composite Bodies**

Parts A and B are slender homogeneous rods weighing 2 kilograms apiece. C is a cylinder weighing 15 kilograms.

Determine $I_A$ using the table below:

<table>
<thead>
<tr>
<th>Body</th>
<th>Mass</th>
<th>$y_{GP}$</th>
<th>$Mx_{GP}$</th>
<th>$Mx_{GP}^2$</th>
<th>$I_{AGP}$</th>
<th>$I_{AP}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>$\frac{1}{2}$</td>
<td>8</td>
<td>4</td>
<td>32</td>
<td>$\frac{4}{5}$</td>
<td>32.8</td>
</tr>
<tr>
<td>B</td>
<td>$\frac{1}{8}$</td>
<td>3</td>
<td>$\frac{3}{8}$</td>
<td>$\frac{9}{8}$</td>
<td>$\frac{3}{8}$</td>
<td>1.5</td>
</tr>
<tr>
<td>C</td>
<td>$\frac{1}{4}$</td>
<td>1</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{4}$</td>
<td>$\frac{1}{10}$</td>
<td>0.35</td>
</tr>
</tbody>
</table>

Locate the mass center of the composite body, remembering that

$$x_G = \frac{\sum M_G x_G}{\sum M}$$
From the preceding frame you know that for the body shown, $I_A = 0.983 \text{ kg-m}^2$ and $x_G = 0.208 \text{ m}$.

Using the relationship

$$I_{A_{\text{total}}} = I_{AG_{\text{total}}} + Md^2$$

find $I_{AG}$ for the composite body.

$$I_{AG1} = \underline{\hspace{2cm}}$$
Correct response to preceding frame

\[ I_{AG} = 0.161 \text{ kg-m}^2 \]

**Solution:**

\[
0.983 = I_{AG} + 19(0.208)^2
\]

\[
I_{AG} = 0.983 - 0.822
\]

Frame 30-39

**Review**

Work Problem 30-1 in your notebook.
Correct response to preceding frame

\[ I_{AG} = 0.053 \text{ slug-ft}^2 \]

<table>
<thead>
<tr>
<th>Part</th>
<th>Mass</th>
<th>( x_{GP} )</th>
<th>( Mx_{GP} )</th>
<th>( Mx_{GP}^2 )</th>
<th>( I_{AGP} )</th>
<th>( I_A )</th>
</tr>
</thead>
<tbody>
<tr>
<td>B</td>
<td>( \frac{1}{4} )</td>
<td>1.5</td>
<td>( \frac{3}{8} )</td>
<td>( \frac{9}{16} )</td>
<td>( \frac{1}{4} ) ( Mr^2 ) + ( \frac{1}{12} ) ( Mh^2 ) = ( \frac{1}{576} ) + ( \frac{1}{192} )</td>
<td>.569</td>
</tr>
<tr>
<td>H</td>
<td>( \frac{1}{16} )</td>
<td>( \frac{2}{3} )</td>
<td>( \frac{2}{48} )</td>
<td>( \frac{4}{144} )</td>
<td>( \frac{1}{12} ) ( Mh^2 ) = ( \frac{1}{12} ) ( \frac{1}{16} \left( \frac{16}{12} \right)^2 ) = ( \frac{16}{1728} )</td>
<td>.037</td>
</tr>
<tr>
<td>Total</td>
<td>( \frac{5}{16} )</td>
<td></td>
<td>( \frac{20}{48} )</td>
<td></td>
<td></td>
<td>.606</td>
</tr>
</tbody>
</table>

\[ x_G = \frac{4}{3} \text{ ft} \]

\[ I_A = .606 \text{ slug-ft}^2 \]

\[ I_{AG} = I_A - Mx_G^2 = .606 - .553 \]

Frame 30-40

Transition

As you have just seen, the actual computation of the moment of inertia for a composite body may involve some rather messy and time consuming arithmetic. It’s easy enough to understand, but it can be tedious to do.

As a result of such computational complexities, we often avoid them completely by making use of a concept known as the radius of gyration; which is merely a short-hand way of specifying the moment of inertia.

The remainder of this unit will be devoted to a brief discussion of the radius of gyration and will take about 15 minutes. Unless you are very tired, or have failed to take a break previously, go on to the next frame.
Radius of Gyration

The radius of gyration of a mass about any axis is the distance from that axis to the point at which the mass could be concentrated without changing the moment of inertia.

If we express the moment of inertia of a mass in terms of its radius of gyration, $k$, we find that $I_A = Mk^2$

For the body shown, $I_A = 34.65 \text{ slug-ft}^2$ and $M = .875 \text{ slug}$.

Find the radius of gyration of the body with respect to the $A$ axis.
Radius of Gyration

Compute the radius of gyration for the following bodies.

(1.)

\[ I_z = 45 \text{ kg-m}^2 \]
\[ M = 3 \text{ kg} \]

The radius of gyration with respect to the ___ axis is _____________

(2.)

\[ I_G = \frac{3}{10} Mr^2 \]
\[ k_G = \underline{\text{______________________}} \]
Radius of Gyration

Is it possible to calculate the radius of gyration for a body whose moment of inertia cannot be found?

[ ] Yes [ ] No

If "no", why not? ________________________________

If "yes", how? ________________________________
Correct response to preceding frame

No, since the radius of gyration is \( k_G = \sqrt{\frac{I}{M}} \) an expression for I must be written before \( k \) can be calculated.

Frame 30-44

**Radius of Gyration**

Initially the moment of inertia and radius of gyration of a body must be computed from the dimensions of the body. Thereafter the moment of inertia may be computed from a knowledge of the mass and the radius of gyration exclusively. Absolutely no information about dimensions will be needed, nor obtained.

An analogy may be drawn here to a dehydrated onion. Onion may be recreated by the addition of water, only if onion existed in the first place to be dehydrated. Similarly, if anyone is to obtain the moment of inertia from the radius of gyration, someone had to know the moment of inertia in the first place.

Go to the next frame.
Frame 30-45

**Radius of Gyration**

Compute the following moments of inertia.

1

(1.)

\[ M = 5 \text{ kilograms} \]

\[ k_z = 0.5 \text{ meters} \]

\[ I_z = \_\_\_\_\_\_\_ \]

(2.)

Total weight = 20 pounds

\[ k_z = 18 \text{ inches} \]

\[ I_z = \_\_\_\_\_\_\_ \]
Correct response to preceding frame

1. \( I_z = \frac{5}{4} \text{ kg-m}^2 \)

2. \( I_z = 1.41 \text{ slug-ft}^2 \)

\[ I_z = \frac{Mk_z}{2} \cdot \frac{20}{32} (2.25) \]

(Slug-ft\(^2\) are far more commonly used in the engineering system than slug-in\(^2\).)

---

Frame 30-46

**Radius of Gyration**

Moments of inertia found from a radius of gyration may be used in the parallel axis theorem in the same way as any other moment of inertia.

The unbalanced wheel weighs 64 pounds and \( k_G = 1.5 \) feet.

Find \( I_A \).

\[ I_A = \text{____________________________} \]
Correct response to preceding frame

\[ I_A = 24.5 \text{ slug-ft}^2 \]

**Solution:**

\[ I_A = Mk_G^2 + Md^2 \]
\[ = 2(2.25) + 2(1^2 + 3^2) \]
\[ = 4.5 + 20 \]

---

Frame 30-47

**Radius of Gyration**

For which of the following bodies can you find \( I_A \) from the given data? If you cannot find it, tell why. If you can, do so!

(1.)

\[ M = 2 \text{ slug} \]
\[ k_B = 1 \text{ foot} \]
\[ r = 1.5 \text{ feet} \]

Can you find \( I_A \)?  \( \square \) Yes  \( \square \) No

Explain:  

(2.)

\[ M = 2 \text{ slug} \]
\[ k_B = 1 \text{ foot} \]
\[ r = 1.5 \text{ feet} \]

Can you find \( I_A \)?  \( \square \) Yes  \( \square \) No

Explain:  

Correct response to preceding frame

(1.) Yes \( I_A = 2(1^2 + 1.5^2) = 6.5 \text{ slug-ft}^2 \)

(2.) No; leastwise, not directly, because neither the \( A \) nor the \( B \) axis passes through the mass center. The parallel axis theorem is therefore not applicable. (You could, of course, get \( I_G \) from \( I_B \) and then \( I_A \) from \( I_G \).)

Frame 30-48

**Radius of Gyration**

When dealing with cylindrical or spherical bodies, care must be taken to avoid confusing radius and radius of gyration.

1. If the disk shown has a mass of 3 slugs and a radius of 2 feet, compute its moment of inertia about the \( z \)-axis passing through its mass center.

   \[ I_z = \]  

2. If the disk has a mass of 3 slugs and a radius of gyration about the \( z \)-axis of 2 feet, compute its moment of inertia about the \( z \) axis.

   \[ I_z = \]  

3. Are the two moments of inertia you just computed the same?

   [ ] Yes  [ ] No
Correct response to preceding frame

1. $I_z = 6 \text{ slug-ft}^2$
   \[ I_z = \frac{1}{2} Mr^2 = \frac{1}{2} (3)(4) \]

2. $I_z = 12 \text{ slug-ft}^2$
   \[ I_z = Mk^2 = 3 (4) \]

3. No

---

Frame 30-49

**Notebook**

Complete page 30-4 and work Problems 30-2 and 30-3 in your notebook.
"Guestimating" Radius of Gyration and Moment of Intertia

As "engineering scientists" we see that we need to know the moment of inertia to find the radius of gyration of a mass, or area.

As "practical engineers" needing a quick estimate of a moment of inertia to estimate a load or acceleration, sometimes an "eyeball" estimate of a radius of gyration can be made.

Suppose we have a steel ring with an outer diameter of 2 meters, an inner diameter of 1.75 meters, and a mass of 30 kilograms:

Since we know that the radius of gyration is sort of an average distance from the mass center --

We know it has to be more than ___________.

We also know that it has to be less than ___________.
Correct response to preceding frame

\[ k_G \text{ is more than } 1.75 / 2 = 0.88 \text{ meters} \]
\[ k_G \text{ is less than } 2.00 / 2 = 1.00 \text{ meters} \]

Frame 30-51

"Guestimating" Radius of Gyration and Moment of Inertia

Using the high-low values above and the mass of 50 kilograms, what would you "guess" the moment of inertia of the ring is about its mass center?

\[ I_G \text{ is about } \underline{\quad} \]

What would you "guess" the moment of inertia of the ring is about point \( I_C \), where the ring contacts the ground?

\[ I_{IC} \text{ is about } \underline{\quad} \]
Correct response to preceding frame

\[ I_G \text{ is about } 45 \text{ kg-m}^2 \]
\[ I_{1C} \text{ is about } 95 \text{ kg-m}^2 \]

Frame 30-52

"Guesstimating" Radius of Gyration and Moment of Intertia

The reason I've included this bit about "guesstimation" is because you may have wondered why we introduced something which appeared to be, at best, a bit useless, and at worst a stupid waste of time.

Actually, the estimation of moment of inertia from an estimated radius of gyration has two practical applications:

First, it may let us make a quick estimate for a "back of the envelope" performance estimate.

Second, it can be a way of making a rough check of a calculation.

Go to the next frame.
Frame 30-53

**Summary**

Hey, guess what, you have just finished the unit! Yes, believe it or not, you now have at your disposal all of the theory necessary to handle any problem which involves a Moment-of-Inertia-of-Mass.

As you progressed through this unit, you may at times have been more than a little disconcerted by the labyrinth of computations required. You will find, however, that the ability to determine the moment of inertia of a mass quickly and accurately is well worth the effort expended.

Since this topic also has numerous important applications in other courses, you will discover that you can reduce your time expenditure in these areas by merely recalling the cut-and-dried procedures you have learned here. After that, the only question will be what you're going to do with all the leisure time you'll have.